

## On Euler numbers, Hilbert sums, Lobachevskii integrals, and their asymptotics

*Dedicated to Tom Koornwinder on the occasion of his 60<sup>th</sup> birthday*

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### 1. INTRODUCTION.

The numbers mentioned in the title (see the formulae (5), (6) and (7) below for definitions) depend on an  $n \in \mathbf{N}$  and can be associated with a central  $B$ -spline, specifically, the  $n$ -fold convolution of the characteristic function of the interval  $[-\frac{1}{2}, \frac{1}{2}]$  with itself. Therefore, their asymptotic behavior for  $n \rightarrow \infty$  is an immediate consequence of the central limit theorem. Furthermore, define

$$(1) \quad I(n, m) = \int_0^\infty \frac{\sin^n \lambda}{\lambda^m} d\lambda \quad (n, m \in \mathbf{N}, n - m \geq 0).$$

The asymptotics of the  $I(n, n - k)$ , for  $n \rightarrow \infty$  and  $k \in \{0\} \cup \mathbf{N}$ , is determined and the  $I(n, m)$ , for all values of  $n$  and  $m$ , are explicitly computed by means of (rudimentary) distribution theory. For  $n - m$  odd, these formulae seem to occur neither in the standard tables of integrals or integral transforms, nor in computer algebra packages like Mathematica (for large values of  $n$  and  $m$ ).

**Convolution.** Denote by  $\chi$  the characteristic function of the interval  $[-\frac{1}{2}, \frac{1}{2}] \subset \mathbf{R}$ , and by  $\chi_n = \chi * \cdots * \chi$  the  $n$ -fold convolution product ( $n \in \mathbf{N}$ ). Then  $\chi_n$  is a spline function, viz. a  $C^{n-2}$ -function on  $\mathbf{R}$  that is piecewise polynomial of degree  $n - 1$ , for  $n \geq 2$ , having  $[-\frac{n}{2}, \frac{n}{2}]$  as its support. Indeed, let  $\chi_n^{(i)}$  ( $0 \leq i \leq n$ ) be the  $i$ -th derivative of  $\chi_n$ . We have the equality  $\chi^{(1)} = \delta_{-\frac{1}{2}} - \delta_{\frac{1}{2}}$  of distributions on  $\mathbf{R}$ , where  $\delta_x$  denotes the Dirac measure at  $x \in \mathbf{R}$ ; whence

$$(2) \quad \chi_n^{(n)} = \chi^{(1)} * \dots * \chi^{(1)} = \sum_{0 \leq j \leq n} (-1)^j \binom{n}{j} \delta_{-\frac{n}{2}+j}.$$

By  $(n-i)$ -fold integration, requiring the continuity of the  $\chi_n^{(i)}$  for  $0 \leq i \leq n-2$  and the vanishing of  $\chi_n^{(n-1)}$  on  $(-\infty, -\frac{n}{2})$ , we find

$$(3) \quad \chi_n^{(i)}(x) = \frac{1}{(n-1-i)!} \sum_{0 \leq j \leq \lfloor x+\frac{n}{2} \rfloor} (-1)^j \binom{n}{j} \left(x + \frac{n}{2} - j\right)^{n-1-i} \quad (0 \leq i \leq n-1).$$

Furthermore, observe that

$$(4) \quad \chi_n^{(i)} = \chi_{n-i} * \chi_i^{(i)} = \chi_{n-i} * \sum_{0 \leq j \leq i} (-1)^j \binom{i}{j} \delta_{-\frac{i}{2}+j}.$$

In particular, we find the Euler numbers  $A(n-1, k)$  (cf. [1, OO pp. 419-421])

$$(5) \quad (n-1)! \chi_n\left(\frac{n}{2} - k\right) = A(n-1, k) := \sum_{0 \leq j \leq k} (-1)^j \binom{n}{j} (k-j)^{n-1} \\ (k \in \{0\} \cup \mathbf{N}).$$

$\chi_n$  is an even function; hence  $\chi_n^{(2i+1)}(0) = 0$ , for  $0 \leq 2i \leq n-3$ . Therefore (cf. [7, Lemma 3.4.7])

$$\sum_{0 \leq j \leq \lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{j} \left(\frac{n}{2} - j\right)^{n-2-2i} = 0 \quad (3 \leq n, 0 \leq 2i \leq n-3).$$

And, taking (3) with  $i = 2$  and  $x = 0$ , we have the following sums, which Hilbert introduced in his work on invariant theory (cf. [4, §9]):

$$(6) \quad H(n) := \sum_{0 \leq j \leq \lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{j} \left(\frac{n}{2} - j\right)^{n-3} = (n-3)! \chi_n^{(2)}(0) \quad (3 \leq n).$$

Notice that (4), with  $i = 2$ , implies  $H(n) = A(n-3, \frac{n}{2}-2) - 2A(n-3, \frac{n}{2}-1) + A(n-3, \frac{n}{2})$ , for  $n$  even.

**Fourier inversion.** The Fourier transform of  $\chi_n$  is  $\lambda \mapsto ((\sin \frac{1}{2}\lambda)/\frac{1}{2}\lambda)^n$ . Using Fourier inversion we get the integrals of Lobachevskĭ (cf. [5, p. 170])

$$(7) \quad \int_0^\infty \frac{\sin^n \lambda}{\lambda^{n-2i}} \cos(2x\lambda) d\lambda = (-1)^i \frac{\pi}{2^{2i+1}} \chi_n^{(2i)}(x) \quad (2 \leq n, 2 \leq n-2i, x \in \mathbf{R}).$$

In particular, with  $n = 2, i = 0, x = 0$ , it follows that

$$(8) \quad \frac{1}{\pi} \int_0^\infty \frac{1 - \cos(2x\lambda)}{x^2} dx = \frac{2}{\pi} \int_0^\infty \frac{\sin^2(x\lambda)}{x^2} dx = \lambda.$$

Furthermore, we obtain from (7)

$$\frac{1}{\pi} \int_0^\infty \frac{\sin^n \lambda}{\lambda^{n-2i}} \frac{\cos(2x\lambda) - 1}{x^2} d\lambda = \frac{(-1)^i}{2^{2i+1}} \frac{1}{x^2} \left( \chi_n^{(2i)}(x) - \chi_n^{(2i)}(0) \right).$$

Integrate this identity with respect to  $x$  from 0 to  $\infty$ , interchange the order of integration and use (8) and the evenness of  $\chi_n^{(2i)}$ . Then it follows that, in the notation of (1),

$$(9) \quad I(n, n-1-2i) = \frac{(-1)^{i+1}}{2^{2i+1}} \int_{-\infty}^0 \frac{1}{x^2} \left( \chi_n^{(2i)}(x) - \chi_n^{(2i)}(0) \right) dx$$

$$(2 \leq n, 2 \leq n-1-2i).$$

As to the restriction on  $i$ , notice that  $\chi_n^{(2i)}$  is piecewise polynomial of degree  $n-1-2i$ . Suppose that  $2i = n-2$ . Then we see from (3) that  $\chi_n^{(2i)} - \chi_n^{(2i)}(0)$  is piecewise linear and nonvanishing on  $[-1, 0]$ . Therefore the RHS, as is the LHS, in (9) is divergent in this case.

**Asymptotics.** The asymptotics of these sums and integrals is a direct consequence of the local form of the central limit theorem (cf. [6, Thm. VIII.2.1]). In our situation, this comes down to the following. Because  $\int_{-\infty}^{\infty} \chi(x) dx = 1$ ,  $\int_{-\infty}^{\infty} x\chi(x) dx = 0$  and  $\int_{-\infty}^{\infty} x^2\chi(x) dx = \frac{1}{12}$ , we have

$$\lim_{n \rightarrow \infty} \left( \frac{n}{12} \right)^{\frac{1}{2}} \chi_n \left( x \sqrt{\frac{n}{12}} \right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad (x \in \mathbf{R}).$$

Even stronger, if we write  $H_p(x)$  for the  $p$ -th Hermite polynomial  $(-1)^p e^{x^2} E^{(p)}(x)$  where  $E(x) = e^{-x^2}$ , it follows that

$$(10) \quad \lim_{n \rightarrow \infty} \left( \frac{n}{6} \right)^{i+\frac{1}{2}} \chi_n^{(2i)} \left( x \sqrt{\frac{n}{6}} \right) = \frac{1}{\sqrt{\pi}} H_{2i}(x) e^{-x^2} \quad (i \in \{0\} \cup \mathbf{N}, x \in \mathbf{R}).$$

In particular, for the Euler numbers we get, if  $x \in \mathbf{R}$  and  $\frac{n}{2} - x\sqrt{n} \in \mathbf{N}$ ,

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{(n-1)!} A \left( n-1, \frac{n}{2} - x\sqrt{n} \right) = \sqrt{\frac{6}{\pi}} e^{-6x^2}.$$

Moreover, from (10) with  $x = 0$  we get Springer's result for the Hilbert sums (cf. [7, Lemma 3.4.7])

$$\lim_{n \rightarrow \infty} \left( \frac{n}{12} \right)^{i+\frac{1}{2}} \frac{1}{(n-1-2i)!} \sum_{0 \leq j \leq \frac{n}{2}} (-1)^j \binom{n}{j} \left( \frac{n}{2} - j \right)^{n-1-2i} = (-1)^i \frac{1}{\sqrt{2\pi}} (2i-1)!!,$$

with the notation  $(2i-1)!! = (2i-1)(2i-3) \cdots 3 \cdot 1$ ; and furthermore (cf. [7, Lemma 3.4.11])

$$(11) \quad \lim_{n \rightarrow \infty} \left( \frac{n}{3} \right)^{i+\frac{1}{2}} I(n, n-2i) = \sqrt{\frac{\pi}{2}} (2i-1)!!.$$

Actually, we have the following generalization of (11):

$$(12) \quad \lim_{n \rightarrow \infty} \left( \frac{n}{6} \right)^{\frac{k+1}{2}} I(n, n-k) = \frac{1}{2} \Gamma \left( \frac{k+1}{2} \right) \quad (k \in \{0\} \cup \mathbf{N}).$$

**Proof of (12).** Obviously the case of  $k = 2i$  is covered by (11). Now suppose  $k = 2i + 1$ . From (9) we obtain

$$I(n, n-1-2i) = \frac{(-1)^{i+1}}{2^{2i+1}} \left(\frac{n}{6}\right)^{-\frac{1}{2}} \int_0^\infty \frac{1}{x^2} \left(\chi_n^{(2i)}(x\sqrt{\frac{n}{6}}) - \chi_n^{(2i)}(0)\right) dx$$

$$(0 \leq 2i \leq n-3).$$

But this equality, taken in conjunction with (10), implies

$$(13) \quad \lim_{n \rightarrow \infty} \left(\frac{n}{6}\right)^{i+1} I(n, n-1-2i) = \frac{(-1)^{i+1}}{2^{2i+1}} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{x^2} (H_{2i}(x)e^{-x^2} - H_{2i}(0)) dx.$$

Using induction over  $i$ , integration by parts, the recurrences  $H_{2i}^{(1)} = 4iH_{2i-1}$  and  $H_{2i+1}(x) = 2xH_{2i}(x) - 4iH_{2i-1}(x)$ , and the orthogonality relations for the Hermite polynomials one easily verifies the following formulae:

$$\int_0^\infty \frac{1}{x^2} (H_{2i}(x)e^{-x^2} - H_{2i}(0)) dx = (-1)^{i+1} 4^i i! \sqrt{\pi} \quad (i \in \{0\} \cup \mathbf{N}),$$

$$\int_0^\infty \frac{1}{x} H_{2i-1}(x)e^{-x^2} dx = (-1)^{i-1} 4^{i-1} (i-1)! \sqrt{\pi} \quad (i \in \mathbf{N}).$$

Therefore the RHS in (13) becomes  $\frac{1}{2}\Gamma(i+1)$ ; that is, (12) is also valid for  $k = 2i + 1$ .

**Evaluation of the integrals  $I(n, m)$ .** These have the following values (see [2] and [3] for a completely different derivation).

(i) Case of  $n - m$  even:

$$(i.a) \quad m = 1, \quad I(2k+1, 1) = \frac{\pi}{2^{k+1}} \frac{(2k-1)!!}{k!} \quad (k \in \{0\} \cup \mathbf{N}),$$

$$(i.b) \quad m = 2, \quad I(2k, 2) = \frac{\pi}{2^k} \frac{(2k-3)!!}{(k-1)!} \quad (k \in \mathbf{N}),$$

$$(i.c) \quad m > 2, \quad I(n, m) = \frac{\pi(-1)^{\frac{n-m}{2}}}{2^{n-m+1}(m-1)!} \sum_{0 \leq j \leq \frac{n-1}{2}} (-1)^j \binom{n}{j} \left(\frac{n}{2} - j\right)^{m-1}.$$

(ii) Case of  $n - m$  odd:

$$(ii.a) \quad m = 1, \quad I(2k, 1) = \infty \quad (k \in \mathbf{N}),$$

$$(ii.b) \quad m > 1, n \text{ even}, \quad I(n, m) = \frac{(-1)^{\frac{n-m-1}{2}}}{2^{n-m}(m-1)!} \sum_{0 \leq j \leq \frac{n-1}{2}} (-1)^j \binom{n}{j} \left(\frac{n}{2} - j\right)^{m-1} \log\left(\frac{n}{2} - j\right),$$

$$(ii.c) \quad m > 1, n \text{ odd}, \quad I(n, m) = \frac{(-1)^{\frac{n-m-1}{2}}}{2^{n-1}(m-1)!} \sum_{0 \leq j \leq \frac{n-1}{2}} (-1)^j \binom{n}{j} (n-2j)^{m-1} \log(n-2j).$$

**Proof of (i).** For (i.a), notice that  $\chi_{2k+1}$  is a polynomial of degree  $2k$  on  $[-\frac{1}{2}, \frac{1}{2}]$ . Hence in (7) we are allowed to replace  $n$  by  $2k+1$  and to choose  $i = k$  and  $x = 0$ . We simplify the answer using the binomial identities. Now (i.b). Simi-

larly we can replace  $n$  by  $2k$ , and choose  $i = k - 1$  and  $x = 0$ . Finally (i.c) is immediate from (7).

**Proof of (ii).** Case (ii.a) has been discussed already. In view of (9) we compute

$$\begin{aligned} & \int_{-\infty}^0 \frac{1}{x^2} \left( \chi_n^{(n-m-1)}(x) - \chi_n^{(n-m-1)}(0) \right) dx \\ &= -\frac{2}{n} \chi_n^{(n-m-1)}(0) + \int_{-\frac{n}{2}}^0 \frac{1}{x^2} \left( \chi_n^{(n-m-1)}(x) - \chi_n^{(n-m-1)}(0) \right) dx. \end{aligned}$$

Integrate by parts twice and use that  $\chi_n^{(n-m)}(0) = \chi_n^{(n-m-1)}(-\frac{n}{2}) = \chi_n^{(n-m)}(-\frac{n}{2}) = 0$ . The integral above takes the form  $-\int_{-\frac{n}{2}}^0 \chi_n^{(n-m+1)}(x) \log(-x) dx$ . Next, denote the  $p$ -fold antiderivative of  $\log$  by  $\log^{[p]}$ ; then there exist constants  $c(p) \in \mathbf{R}$  such that  $\log^{[p]}(-x) = \frac{1}{p!} x^p \log(-x) + c(p) x^p$ . In particular,  $\log^{[p]}(0) = 0$ , for  $p \in \mathbf{N}$ . Since  $\chi_n^{(i)}(-\frac{n}{2}) = 0$ , for  $0 \leq i \leq n-2$ , and the resulting integrals are convergent, we can continue integrating by parts, viz.  $m-2$  many times; and we obtain  $(-1)^{m-1} \int_{-\frac{n}{2}}^0 \chi_n^{(n-1)}(x) \log^{[m-2]}(-x) dx$ . Since  $-\frac{n}{2} + [\frac{n-1}{2}] = -1$  or  $-\frac{1}{2}$ , we get in view of (2) after one more integration by parts

$$\begin{aligned} & (-1)^m \sum_{0 \leq j \leq [\frac{n-1}{2}]} (-1)^j \binom{n}{j} \log^{[m-1]} \left( -\left( -\frac{n}{2} + j \right) \right) \\ &= \frac{-1}{(m-1)!} \sum_{0 \leq j \leq [\frac{n-1}{2}]} (-1)^j \binom{n}{j} \left( \frac{n}{2} - j \right)^{m-1} \log \left( \frac{n}{2} - j \right) - c(m-1) \sum_{0 \leq j \leq [\frac{n-1}{2}]} (-1)^j \binom{n}{j} \left( \frac{n}{2} - j \right)^{m-1}. \end{aligned}$$

But the last sum is equal to  $(m-1)! \chi_n^{(n-m)}(0) = 0$ . The computation of the  $I(n, m)$  now can be completed by combining the results above.

**Remarks.** The formulae in the section above also can be obtained by means of complex analysis. The author is grateful to J.J. Duistermaat and T.A. Springer for discussions concerning this manuscript.

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